

Another viewpoint on \mathcal{J} -spaces

Markus Spitzweck

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Abstract

We give an interpretation of \mathcal{J} -spaces in terms of symmetric spectra in symmetric sequences. As application we show how one can define graded endomorphism objects in a general situation. As example we discuss the motivic bigraded endomorphisms of a motivic E_∞ -ring spectrum. Finally we give an ∞ -categorical interpretation of our result.

1 Introduction

The category of \mathcal{J} -spaces has been introduced in [7] to figure as a suitable target for the graded units of an E_∞ -ring spectrum. Recall the units $\mathrm{GL}_1(\mathbf{E})$ of a fibrant E_∞ -ring spectrum \mathbf{E} are defined to be $\Omega^\infty(\mathbf{E})^\times$, i.e. those connected components of $\Omega^\infty(\mathbf{E})$ which are invertible components for the E_∞ -structure on $\Omega^\infty(\mathbf{E})$ coming from that one on \mathbf{E} .

Equivalently one can view $\mathrm{GL}_1(\mathbf{E})$ as the space of \mathbf{E} -module automorphisms from \mathbf{E} to itself. This concept does not capture the possibility of graded automorphisms of \mathbf{E} , i.e. self-equivalences from a (positive or negative) suspension of \mathbf{E} to \mathbf{E} .

In particular the canonical map $\mathrm{GL}_1(\mathbf{e}) \rightarrow \mathrm{GL}_1(\mathbf{E})$ is always an equivalence, where $\mathbf{e} \rightarrow \mathbf{E}$ is the connected cover.

The need for some graded version of $\mathrm{GL}_1(\mathbf{E})$ stems in particular from the theory of topological logarithmic structures, see [6].

In [7] the graded units of an E_∞ -ring spectrum manifest themselves as \mathcal{J} -spaces. In loc. cit. it is discussed that only in the context of *graded* topological logarithmic structures there is an interesting such structure on the connective cover of the topological complex K -theory spectrum coming from the Bott element.

In this note we discuss \mathcal{J} -objects in general (symmetric monoidal) categories and model categories. Our main result (Proposition 2.1) says that the category of \mathcal{J} -objects is equivalent to that one of symmetric T -spectra, where T is a particular symmetric

sequence. In the proof we introduce as intermediate step a structure which we call T -data. The author was informed that these data were in fact the first manifestation of \mathcal{J} -spaces Sagave-Schlichtkrull used.

Our applications are mainly for motivic E_∞ -spectra, we show how to obtain bigraded versions of the motivic endomorphisms of such a spectrum.

We also discuss the ∞ -categorical content of our result.

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2 \mathcal{J} -spaces as symmetric spectra

For any natural number n we set $\underline{n} = \{1, \dots, n\}$. As in [7] we denote by \mathcal{J} the following category: objects are pairs $(\underline{m}, \underline{n})$, morphisms from $(\underline{m}, \underline{n})$ to $(\underline{k}, \underline{l})$ are triples (φ, ψ, α) where $\varphi: \underline{m} \rightarrow \underline{k}$ and $\psi: \underline{n} \rightarrow \underline{l}$ are injections and $\alpha: (\underline{k} - \varphi(\underline{m})) \rightarrow (\underline{l} - \psi(\underline{n}))$ is a bijection. Composition takes as bijection the disjoint union of the induced bijections. \mathcal{J} is symmetric monoidal, where on objects the tensor product is concatenation. The symmetry isomorphism involves the shuffle permutation.

Let \mathcal{C} be a symmetric monoidal category with all colimits. We equip the category of symmetric sequences \mathcal{C}^Σ and the category of \mathcal{J} -objects $\mathcal{C}^\mathcal{J}$ with the Day convolution tensor product. We let $T = T_\mathcal{C} = (\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots)$ be the symmetric sequence in \mathcal{C} where the tensor unit sits in degree one, and in every other degree the initial object. We let $\mathbf{Spt}_T^\Sigma(\mathcal{C}^\Sigma)$ be the symmetric monoidal category of symmetric T -spectra in \mathcal{C}^Σ . By definition this is the category of modules over the commutative monoid $\mathrm{Sym}(T) = (\mathbf{1}, T, T^{\otimes 2}, T^{\otimes 3}, \dots)$ in $(\mathcal{C}^\Sigma)^\Sigma$.

Proposition 2.1: *The categories $\mathcal{C}^\mathcal{J}$ and $\mathbf{Spt}_T^\Sigma(\mathcal{C}^\Sigma)$ are naturally equivalent as symmetric monoidal categories.*

Proof. An object in $\mathbf{Spt}_T^\Sigma(\mathcal{C}^\Sigma)$ is a symmetric sequence (X_0, X_1, \dots) in \mathcal{C}^Σ together with bonding maps $X_i \otimes T \rightarrow X_{i+1}$ such that the iterates

$$X_i \otimes T^{\otimes p} \rightarrow X_{i+p}$$

are $\Sigma_i \times \Sigma_p$ -equivariant.

The functor $X \mapsto X \otimes T$ on \mathcal{C}^Σ has a right adjoint, denoted $X \mapsto X^T$. We have $(X^T)_n = X_{n+1}$ with the Σ_n -action induced by the Σ_{n+1} -action on X_{n+1} via the natural inclusion $\Sigma_n \rightarrow \Sigma_{n+1}$.

Thus the bonding map $X_i \otimes T \rightarrow X_{i+1}$ of a T -spectrum X is adjoint to a map $X_i \rightarrow X_{i+1}^T$. Such a map is given by a family of maps $\varphi_{i,n}: X_{i,n} \rightarrow X_{i+1,n+1}$, such that the n -th map is Σ_n -equivariant. Moreover, the equivariance of the iterated bonding maps translates to the following statement: the composition

$$\Phi_{i,n,p} := \varphi_{i+p-1,n+p-1} \circ \cdots \circ \varphi_{i,n}: X_{i,n} \rightarrow X_{i+p,n+p}$$

is Σ_i -equivariant and for each $g \in \Sigma_p$ we have $g \circ \Phi_{i,n,p} = \Phi_{i,n,p}$. Here g acts on $X_{i+p,n+p}$ via the inclusion $\iota_{i,n,p}: \Sigma_p \rightarrow \Sigma_{i+p} \times \Sigma_{n+p}$ which is the product of the maps from Σ_p into Σ_{i+p} and Σ_{n+p} which permute the last p elements.

So a symmetric T -spectrum amounts to the following data:

objects $X_{i,n} \in \mathcal{C}$ with a $\Sigma_i \times \Sigma_n$ -action, $\Sigma_i \times \Sigma_n$ -equivariant maps $\varphi_{i,n}: X_{i,n} \rightarrow X_{i+1,n+1}$ such that their iterates $\Phi_{i,n,p}: X_{i,n} \rightarrow X_{i+p,n+p}$ obey the condition $g \circ \Phi_{i,n,p} = \Phi_{i,n,p}$ for all $g \in \Sigma_p$ (where we use the above inclusion $\iota_{i,n,p}$). We call such a datum a T -datum. T -data form a category equivalent to T -spectra.

We have to see that a T -datum is equivalent to a functor $\mathcal{J} \rightarrow \mathcal{C}$.

Let be given a T -datum (in the notation as above). Of course it is clear what the functor $\mathcal{J} \rightarrow \mathcal{C}$ should do on objects, it should send $(\underline{i}, \underline{n})$ to $X_{i,n}$.

Let $(\varphi, \psi, \alpha): (\underline{i}, \underline{n}) \rightarrow (\underline{i} \sqcup \underline{p}, \underline{n} \sqcup \underline{p})$ be a map in \mathcal{J} . We denote by $\Psi_{i,n,p}: (\underline{i}, \underline{n}) \rightarrow (\underline{i} \sqcup \underline{p}, \underline{n} \sqcup \underline{p})$ the standard map, i.e. sending \underline{i} to \underline{i} via the identity map, similarly for \underline{n} , and the required bijection is induced from the identity on \underline{p} . Then there is a $(a, b) \in \Sigma_{i+p} \times \Sigma_{n+p}$ such that $(\varphi, \psi, \alpha) = (a, b) \circ \Psi_{i,n,p}$. Moreover such a (a, b) is unique with this property up to precomposition with a $g \in \Sigma_p$ which acts via the inclusion $\iota_{i,n,p}$.

The image of the functor $\mathcal{J} \rightarrow \mathcal{C}$ we want to define on the map (φ, ψ, α) is defined to be $(a, b) \circ \Phi_{i,n,p}$. This is independent of the possible choices for (a, b) because of the last property of a T -datum.

This assignment indeed defines a functor $\mathcal{J} \rightarrow \mathcal{C}$ because of the equivariance of the maps in a T -datum. One extends this to a functor from the category of T -data to the functor category $\mathcal{C}^{\mathcal{J}}$.

On the other hand starting with a functor $f: \mathcal{J} \rightarrow \mathcal{C}$ defines a T -datum by setting $X_{i,n} := f((\underline{i}, \underline{n}))$ with the induced $\Sigma_i \times \Sigma_n$ -action. The maps $X_{i,n} \rightarrow X_{i+1,n+1}$ are defined to be the $f(\Psi_{i,n,1})$. These are clearly $\Sigma_i \times \Sigma_n$ -equivariant. Moreover the additional condition on the Σ_p -invariance follows since in \mathcal{J} we have the identity $g \circ \Psi_{i,n,p} = \Psi_{i,n,p}$ for $g \in \Sigma_p$ acting via the inclusion $\iota_{i,n,p}$.

This assignment extends to a functor from $\mathcal{C}^{\mathcal{J}}$ to T -data.

The two functors defined are clearly inverse to each other.

We have to see that these functors preserve the tensor product.

For that we describe the functor $\mathcal{C}^{\mathcal{J}} \rightarrow \mathbf{Spt}_T^{\Sigma}(\mathcal{C}^{\Sigma})$ in different terms. Let $j: \Sigma^2 \rightarrow \mathcal{J}$ be the embedding. The functor j induces a symmetric monoidal functor $j_!: \mathcal{C}^{\Sigma^2} \rightarrow \mathcal{C}^{\mathcal{J}}$ with right adjoint j^* . This adjunction induces an adjunction $\mathbf{Mod}(j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})) \rightleftarrows \mathcal{C}^{\mathcal{J}}$. The left adjoint of this adjunction is given by the factorization

$$\mathbf{Mod}(j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})) \rightarrow \mathbf{Mod}(j_!(j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})) \rightarrow \mathcal{C}^{\mathcal{J}},$$

where the last functor is given by push forward along the canonical map of commutative monoids $j_!(j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})) \rightarrow \mathbf{1}_{\mathcal{C}^{\mathcal{J}}}$. Thus this left adjoint is also symmetric monoidal.

The unit $\mathbf{1}_{\mathcal{C}^{\mathcal{J}}}$ is given by $\mathrm{Hom}_{\mathcal{J}}((\underline{0}, \underline{0}), -) \times \mathbf{1}_{\mathcal{C}}$. Thus $\mathbf{1}_{\mathcal{C}^{\mathcal{J}}}(\underline{m}, \underline{n}) = \Sigma_n \times \mathbf{1}_{\mathcal{C}}$ if $m = n$ and the initial object otherwise. It is easy to see that $j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})$ is canonically isomorphic to $\mathrm{Sym}(T)$ as commutative monoids. Moreover the canonical functor

$$\mathcal{C}^{\mathcal{J}} \rightarrow \mathbf{Mod}(j^*(\mathbf{1}_{\mathcal{C}^{\mathcal{J}}})) \simeq \mathbf{Mod}(\mathrm{Sym}(T)) = \mathbf{Spt}_T^{\Sigma}(\mathcal{C}^{\Sigma})$$

is seen to be the functor described in the first part of this proof. But we have already seen that its left adjoint is symmetric monoidal. This finishes the proof. \square

Remark 2.2: Suppose \mathcal{C} is a category with (set-indexed) coproducts. Then it is still possible to describe $\mathcal{C}^{\mathcal{J}}$ in terms of symmetric spectra. Namely, $(\mathcal{C}^{\Sigma})^{\Sigma}$ is tensored over $(\mathrm{Set}^{\Sigma})^{\Sigma}$, and $\mathcal{C}^{\mathcal{J}}$ is equivalent to $\mathbf{Mod}_{\mathcal{C}}(\mathrm{Sym}(T_{\mathrm{Set}}))$.

Let now \mathcal{C} and \mathcal{D} be cocomplete symmetric monoidal categories and $f: \mathcal{C} \rightarrow \mathcal{D}$ a symmetric monoidal functor. Suppose f is cocontinuous and the tensor product of \mathcal{D} commutes with colimits separately in each variable. Let $K \in \mathcal{D}$. Then there is a canonical cocontinuous symmetric monoidal functor $f^K: \mathcal{C}^{\Sigma} \rightarrow \mathcal{D}$ which sends $T_{\mathcal{C}}$ to K and prolongs f . In formulas it is given by

$$f^K((X_0, X_1, X_2, \dots)) = \coprod_{n \geq 0} f(X_n) \otimes_{\Sigma_n} K^{\otimes n}.$$

This functor extends to spectra

$$f_{\mathrm{Spt}}^K: \mathcal{C}^{\mathcal{J}} \simeq \mathbf{Spt}_T^{\Sigma}(\mathcal{C}^{\Sigma}) \rightarrow \mathbf{Spt}_K^{\Sigma}(\mathcal{D}).$$

Example 2.3: Let \mathcal{S} be the category of spaces, i.e. either topological spaces or simplicial sets and \mathcal{S}_{\bullet} the category of pointed spaces. Let $f: \mathcal{S} \rightarrow \mathcal{S}_{\bullet}$ be the functor which adds a basepoint. Then the induced functor

$$f_{\mathrm{Spt}}^{\mathcal{S}_{\bullet}}: \mathcal{S}^{\mathcal{J}} \rightarrow \mathbf{Spt}_{\mathcal{S}_{\bullet}}^{\Sigma}(\mathcal{S}_{\bullet})$$

is the functor $\mathbb{S}^{\mathcal{J}}[-]$ of [7].

3 Model structures

We suppose now that \mathcal{C} and \mathcal{D} are left proper cellular or combinatorial symmetric monoidal model categories and $f: \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal left Quillen functor. We suppose the unit in \mathcal{C} is cofibrant and $K \in \mathcal{D}$ is cofibrant. We equip the above mentioned categories of spectra $\mathbf{Spt}_T^{\Sigma}(\mathcal{C}^{\Sigma})$ and $\mathbf{Spt}_K^{\Sigma}(\mathcal{D})$ with the stable model structures of [2]. We have to verify that the required localizations exist in the combinatorial case. Therefore we have to see that the categories $\text{Mod}(\text{Sym}(T))$ and/or $\text{Mod}(\text{Sym}(K))$ are locally presentable. This follows from [4, Corollary 2.3.8.(1)].

By transfer of structure the stable model structure on $\mathbf{Spt}_T^{\Sigma}(\mathcal{C}^{\Sigma})$ induces a model structure on the equivalent category $\mathcal{C}^{\mathcal{J}}$. This is a localization of the projective model structure on $\mathcal{C}^{\mathcal{J}}$, the local objects are those diagrams for which the transition maps are weak equivalences. It follows that in the case of spaces this model structure is the same as the \mathcal{J} -model structure introduced by Sagave-Schlichtkrull [7]. The weak equivalences in this model structure are precisely the maps of diagrams of spaces which induce weak equivalences on homotopy colimits, see loc. cit.

4 Graded endomorphism objects

Let the notation be as in the last section. Let $f_{\text{Spt}}^K: \mathcal{C}^{\mathcal{J}} \rightarrow \mathbf{Spt}_K^{\Sigma}(\mathcal{D})$ be the functor introduced in section 2. We claim this is a left Quillen functor. Indeed, for a discrete group G and a G -object X in \mathcal{D} which is underlying cofibrant the functor $\mathcal{D}[G] \rightarrow \mathcal{D}$, $Y \mapsto Y \otimes_G X$, is a left Quillen functor.

Let \mathcal{O} be an operad in \mathcal{C} , e.g. an E_{∞} -operad (for example \mathcal{C} could be a simplicial symmetric monoidal model category, then we can take the image of an E_{∞} -operad in simplicial sets). By abuse of notation we will also talk about \mathcal{O} -algebras in \mathcal{D} and the categories of symmetric spectra. By this we shall mean algebras over the respective image of \mathcal{O} .

Suppose E is an \mathcal{O} -algebra in $\mathbf{Spt}_K^{\Sigma}(\mathcal{D})$. Let r be the right adjoint to f_{Spt}^K . Then $r(E)$ is also an \mathcal{O} -algebra. If E is underlying fibrant then $r(E)$ has the correct homotopy type. In the case where \mathcal{O} is Σ -cofibrant one can always achieve this by using semi model structures on \mathcal{O} -algebras.

In particular cases one can think about $r(E)$ as a graded endomorphism object of E , e.g. for Example 2.3 and in the motivic situation, see subsection 4.2.

4.1 Grading by tensor invertible objects

We suppose given cofibrant objects $K_1, \dots, K_n \in \mathcal{D}$ such that these are tensor invertible in $\mathrm{Ho}\mathcal{D}$. We let $\mathrm{Spt}_{\underline{K}}^{\Sigma}(\mathcal{D})$ be the model category of symmetric K_1, \dots, K_n -multi-spectra in \mathcal{D} .

By our assumption and [2, Theorem 9.1] the adjunction $\mathcal{D} \rightleftarrows \mathrm{Spt}_{\underline{K}}^{\Sigma}(\mathcal{D})$ is a Quillen equivalence.

By iterating the definition of the functor f_{Spt}^K we get an induced symmetric monoidal left Quillen functor

$$f_{\mathrm{Spt}}^{\underline{K}}: \mathcal{C}^{\mathcal{J}^n} \rightarrow \mathrm{Spt}_{\underline{K}}^{\Sigma}(\mathcal{D}).$$

Suppose that \mathcal{C} is simplicial such that we have good models of E_{∞} -operads.

The derived right adjoint of $f_{\mathrm{Spt}}^{\underline{K}}$ on the level of E_{∞} -algebras induces a functor from the homotopy category of E_{∞} -algebras in \mathcal{D} to the homotopy category of E_{∞} -algebras in $\mathcal{C}^{\mathcal{J}^n}$, which can be thought of as multi-graded E_{∞} -algebras in \mathcal{C} , see section 5.

4.2 The motivic example

We specialize the construction of subsection 4.1 to the motivic situation. Let $\mathcal{S}^{\mathrm{mot}}$ be the model category of \mathbf{P}^1 -spectra for a given base scheme. We let K_1 and K_2 be cofibrant models for the motivic spheres $S^{1,0}$ and $S^{0,1}$. For the category \mathcal{C} we either take simplicial sets with the natural functor f to $\mathcal{S}^{\mathrm{mot}}$ or the category of motivic spaces, i.e simplicial presheaves on smooth schemes over the base scheme with an \mathbb{A}^1 - and Nisnevich-local model structure. The functor f in this case is the \mathbf{P}^1 -suspension functor followed by adding a basepoint.

Let r' be the right adjoint to the functor

$$f_{\mathrm{Spt}}^{K_1, K_2}: \mathcal{C}^{\mathcal{J}^2} \rightarrow \mathrm{Spt}_{K_1, K_2}^{\Sigma}(\mathcal{S}^{\mathrm{mot}})$$

defined in subsection 4.1.

Then the image with respect to r' of a (fibrant) motivic E_{∞} -ring spectrum \mathbf{E} in $\mathcal{C}^{\mathcal{J}^2}$ is a bigraded version of the endomorphism space of \mathbf{E} :

Definition 4.1: *Let \mathbf{E} be a motivic E_{∞} -spectrum and $\mathbf{E} \rightarrow R\mathbf{E}$ a fibrant replacement. Then $r'(R\mathbf{E})$ is defined to be the (derived) bigraded endomorphism space of \mathbf{E} .*

One can extract a bigraded version of $\mathrm{GL}_1\mathbf{E}$ by taking the (sectionwise) invertible endomorphisms (in the case \mathcal{C} is motivic spaces one has to work with a fibrant model).

Remark 4.2: *One feature of \mathcal{J} -spaces is that they allow for a positive (flat) model structure with the property that commutative \mathcal{J} -space monoids carry a model structure, see [7]. It is natural to expect that these features carry over to the motivic setting, so that we can talk about strict commutative algebra objects in $\mathcal{C}^{\mathcal{J}^2}$ instead about E_∞ -algebras. We note that positive flat model structures on motivic symmetric spectra and algebras over arbitrary operads in motivic symmetric spectra have been worked out in [1].*

In the motivic stable homotopy category there are more tensor invertible elements than just the motivic spheres, see [3]. So one may enlarge the number of grading directions for the graded units of a motivic E_∞ -spectrum.

There might be relations among the tensor invertible elements. This will not be reflected in our version of graded endomorphisms and units. We leave this question to future work.

5 The ∞ -categorical interpretation

It is proven in [7] that the classifying space $B\mathcal{J}$ is a model for QS^0 . Suppose for the rest that our model categories are additionally simplicial model categories.

Our framework for ∞ -categories will be mainly the weak Kan complexes resp. quasi-categories, see e.g. [5].

We consider a usual category as an ∞ -category by the nerve construction, by abuse of notation we also write C for the ∞ -category associated to the category C .

We view a topological space as an ∞ -category via the singular simplicial set functor, in particular classifying spaces of categories are viewed in such a way as ∞ -categories.

Let K be a simplicial set and $S \subset K_1$ a subset of the edges. We denote by $K[S^{-1}]$ the pushout

$$\begin{array}{ccc} \coprod_{s \in S} \Delta^1 & \longrightarrow & K \\ \downarrow & & \downarrow \\ \coprod_{s \in S} \overline{\Delta^1} & \longrightarrow & K[S^{-1}] \end{array} \quad ,$$

where $\overline{\Delta^1}$ denotes the nerve of the category with two objects and one isomorphism between these.

We note that the above pushout is a homotopy pushout both in the usual model structure on \mathbf{sSet} and the Joyal model structure. Also $K \rightarrow K[S^{-1}]$ is a usual weak equivalence in \mathbf{sSet} .

We set $K[K^{-1}] = K[K_1^{-1}]$.

Lemma 5.1: *Let C be an ∞ -category, $K \in \mathbf{sSet}$ and $S \subset K_1$. Then the map $C^{K[S^{-1}]} \rightarrow C^K$ is a fully faithful map between ∞ -categories whose essential image consists of those functors $K \rightarrow C$ which send each edge in S to an equivalence.*

Proof. We first prove the special case saying that $C^{\overline{\Delta^1}} \rightarrow C^{\Delta^1}$ is fully faithful with essential image the edges in C which are equivalences. Let $s: \Delta^1 \rightarrow C$ be an equivalence. Then s factors through the maximal Kan complex in C . Thus the lifting property in the usual model structure shows that s can be extended to a map $\bar{s}: \overline{\Delta^1} \rightarrow C$. This shows the claim about the essential image.

Let $f, g: \Delta^1 \rightarrow C$ be equivalences, $f: x \rightarrow y$, $g: w \rightarrow z$. Then the claim about the fully faithfulness follows from the fact that there is a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{C^{\Delta^1}}(f, g) & \longrightarrow & \mathrm{Map}_C(y, z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_C(x, w) & \longrightarrow & \mathrm{Map}_C(x, z) \end{array}$$

in \mathbf{HosSet} where every map is an isomorphism and that $\overline{\Delta^1} \rightarrow \mathrm{pt}$ is a Joyal equivalence (so that the mapping space in $C^{\overline{\Delta^1}}$ can be computed as a mapping space in C).

We prove now the general statement. Since the defining square for $K[S^{-1}]$ is a homotopy pushout square the square

$$\begin{array}{ccc} C^{K[S^{-1}]} & \longrightarrow & \prod_{s \in S} C^{\overline{\Delta^1}} \\ \downarrow & & \downarrow \\ C^K & \longrightarrow & \prod_{s \in S} C^{\Delta^1} \end{array}$$

is a homotopy pullback square in the Joyal model structure. By what we have already proved the right vertical arrow is fully faithful with essential image collections of arrows such that each arrow is a weak equivalence. The claim follows from the fact that homotopy pullbacks of fully faithful maps are fully faithful with essential image those objects which map to the essential image of the given map. \square

Lemma 5.2: *Let $K \in \mathbf{sSet}$, $K[K^{-1}] \rightarrow R$ a Joyal fibrant replacement. Then R is a Kan complex. In particular the map $K \rightarrow R$ is a fibrant replacement in the usual model structure on \mathbf{sSet} .*

Proof. By [5, Proposition 1.2.5.1] an ∞ -category C is a Kan complex if and only if the homotopy category $\mathrm{h}C$ associated to C is a groupoid. So we have to show that $\mathrm{h}R$ is

a groupoid. But by construction $\mathbf{h}R$ is generated as a category by the morphisms and their inverses which come from K . These are all invertible, thus the claim follows. \square

We keep the notation from section 3.

Proposition 5.3: *The local model structure on $\mathcal{C}^{\mathcal{J}}$ models the ∞ -category $\mathcal{C}^{\mathbf{B}\mathcal{J}} \simeq \mathcal{C}^{QS^0}$, where \mathcal{C} is the ∞ -category associated to \mathcal{C} .*

Proof. First observe that the projective model structure on $\mathcal{C}^{\mathcal{J}}$ models the ∞ -category $\mathcal{C}^{\mathcal{J}}$ by the strictification theorem [9, Theorem 4.2.1]. One then checks that the local model structure on $\mathcal{C}^{\mathcal{J}}$ models the full subcategory \mathcal{D} of $\mathcal{C}^{\mathcal{J}}$ consisting of functors which send all maps in \mathcal{J} to equivalences.

Let K be the nerve of \mathcal{J} . Then by Lemma 5.1 the ∞ -categories \mathcal{D} and $\mathcal{C}^{K[K^{-1}]}$ are canonically equivalent.

By Lemma 5.2 $K[K^{-1}]$ and $\mathbf{B}\mathcal{J}$ are canonically equivalent. Thus we finally have $\mathcal{D} \simeq \mathcal{C}^{K[K^{-1}]} \simeq \mathcal{C}^{\mathbf{B}\mathcal{J}} \simeq \mathcal{C}^{QS^0}$. \square

Corollary 5.4: *The symmetric stabilization of the ∞ -category \mathcal{C}^{Σ} with respect to the object $(\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots)$ (where we use the model category description for the symmetric stabilization) is equivalent to the category \mathcal{C}^{QS^0} .*

Proof. This follows from Propositions 2.1 and 5.3. \square

Pretending suitable ∞ -categorical universal properties one may say that this result states that \mathcal{C}^{QS^0} is the universal symmetric monoidal ∞ -category with all colimits and a closed tensor product over \mathcal{C} generated by one tensor invertible object. This universal property should hold in some ∞ -category of such symmetric monoidal ∞ -categories where the morphisms are the symmetric monoidal cocontinuous functors. Note that \mathcal{C}^{Σ} should be the universal cocomplete symmetric monoidal ∞ -category generated by one object. This free object in \mathcal{C}^{Σ} is $(\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots)$.

Let be given a symmetric monoidal cocontinuous functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between cocomplete closed symmetric monoidal ∞ -categories. Let $K \in \mathcal{D}$ be tensor invertible. Suppose \mathcal{C}^{QS^0} satisfies the universal property mentioned in the last paragraph. Let $\mathcal{C}^{QS^0} \rightarrow \mathcal{D}$ be the induced symmetric monoidal functor sending the free tensor invertible object in \mathcal{C}^{QS^0} to K . One may then ask if there exists a factorization $\mathcal{C}^{QS^0} \rightarrow \mathcal{C}^{\mathbb{Z}} \rightarrow \mathcal{D}$. This should express that the permutation actions on the $K^{\otimes n}$ are strictly the identity, at least for the universal case where the functor F is the identity.

Such a factorization for example exists for chain complexes with $K = \mathbb{Z}[2]$ or motives with $K = \mathbb{Z}(1)$ or $K = \mathbb{Z}(1)[2]$. This can be seen by considering strong periodizations of the motivic Eilenberg MacLane spectrum, see [8].

There could also be intermediate factorizations, e.g. through $\mathcal{C}^{l_1(QS^0)}$. Here $l_1(QS^0)$ denotes the space obtained from QS^0 by killing all homotopy groups (of all connected components) above the first.

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Department of Mathematics, University of Oslo, Norway.
e-mail: markussp@math.uio.no